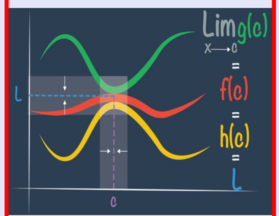
Calculus I

Lecture 12



Feb 19-8:47 AM

Class QZ 9

Find $f'(x)$: **Do not Simplify**

1) $f(x) = (x^2 - 5x + 10)^3$ $\quad f'(x) = 3(x^2 - 5x + 10)^2 \cdot (2x - 5)$

$$\frac{d}{dx}\left[\frac{x}{x+1}\right] = \frac{1(x+1) - x \cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

2) $f(x) = \sin^2\left(\frac{x}{x+1}\right) = \left[\sin\left(\frac{x}{x+1}\right)\right]^2$

$$\boxed{f'(x) = 2 \cdot \sin\frac{x}{x+1} \cdot \cos\frac{x}{x+1} \cdot \frac{1}{(x+1)^2}}$$

Jul 7-7:09 AM

find $f'(x)$

1) $f(x) = x^2\sqrt{x} = x^2 \cdot x^{1/2} = x^{5/2}$

$f'(x) = \frac{5}{2}x^{5/2-1} = \frac{5}{2}x^{3/2} = \frac{5}{2}\sqrt{x^3}$

$= \boxed{\frac{5}{2}x\sqrt{x}}$

2) $f(x) = \dfrac{x^2}{\sqrt{x}} = x^{2-1/2} = x^{3/2}$

$f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{1/2} = \boxed{\frac{3}{2}\sqrt{x}}$

Jul 7-8:18 AM

3) $f(x) = (x^3 - \cos x^2)^4$

$f'(x) = 4(x^3 - \cos x^2)^3 \cdot (3x^2 - (-\sin x^2)\cdot 2x)$

$\boxed{f'(x) = 4(x^3 - \cos x^2)^3(3x^2 + 2x\sin x^2)}$

4) $f(x) = \sqrt{\dfrac{x}{1+\sin x}}$

$f(x) = \left(\dfrac{x}{1+\sin x}\right)^{1/2}$

$f'(x) = \frac{1}{2}\left(\dfrac{x}{1+\sin x}\right)^{-1/2} \cdot \dfrac{1(1+\sin x) - x\cos x}{(1+\sin x)^2}$

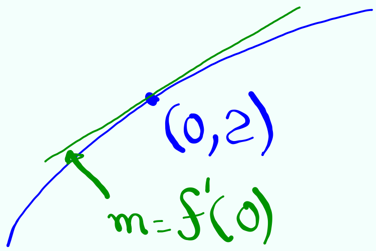
$= \frac{1}{2}\left(\dfrac{1+\sin x}{x}\right)^{1/2} \cdot \dfrac{1+\sin x - x\cos x}{(1+\sin x)^2}$

$= \frac{1}{2} \cdot \dfrac{(1+\sin x)^{\boxed{1/2}}}{x^{1/2}} \cdot \dfrac{1+\sin x - x\cos x}{(1+\sin x)^{\boxed{2}}}$ $\quad 2 - \frac{1}{2} = \frac{3}{2}$

$= \boxed{\dfrac{1+\sin x - x\cos x}{2\sqrt{x}\,(1+\sin x)^{3/2}}}$

Jul 7-8:25 AM

Find the equation of the tan. line to the graph of $f(x)=\sqrt{\dfrac{x+8}{x+2}}$ at $x=0$.



$$f(x)=\left(\frac{x+8}{x+2}\right)^{1/2}$$

$$f'(x)=\frac{1}{2}\left(\frac{x+8}{x+2}\right)^{-1/2}\cdot\frac{1(x+2)-(x+8)\cdot 1}{(x+2)^2}$$

$$f'(x)=\frac{1}{2}\left(\frac{x+2}{x+8}\right)^{1/2}\cdot\frac{-6}{(x+2)^2}$$

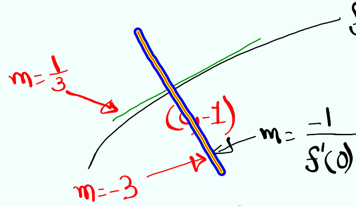
$$f'(0)=\cancel{\frac{1}{2}}\cdot\frac{1}{2}\cdot\frac{\overset{3}{\cancel{-6}}}{4}=\frac{-3}{8}$$

$$y-2=\frac{-3}{8}(x-0)$$

$$\boxed{y=\frac{-3}{8}x+2}$$

Jul 7-8:36 AM

Find eqn of the normal line to the graph of $f(x)=\sqrt[3]{\sin x-\cos x}$ at $x=0$.



$$f(x)=\sqrt[3]{\sin x-\cos x}$$

$$f(x)=(\sin x-\cos x)^{1/3}$$

$$f'(x)=\frac{1}{3}(\sin x-\cos x)^{1/3-1}\cdot(\cos x+\sin x)$$

$$f'(x)=\frac{1}{3}(\sin x-\cos x)^{-2/3}(\cos x+\sin x)$$

$$f'(x)=\frac{\cos x+\sin x}{3\sqrt[3]{(\sin x-\cos x)^2}}$$

$$f'(0)=\frac{\overset{1}{\cancel{\cos 0}}+\overset{0}{\cancel{\sin 0}}}{3\sqrt[3]{(\underset{0}{\cancel{\sin 0}}-\underset{1}{\cancel{\cos 0}})^2}}=\frac{1}{3\sqrt[3]{1}}=\frac{1}{3}$$

$$y-y_1=m(x-x_1)$$

$$y-(-1)=-3(x-0)$$

$$y+1=-3x \qquad \boxed{y=-3x-1}$$

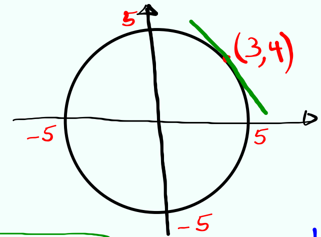
Jul 7-8:44 AM

Given $x^2 + y^2 = 25$

1) Graph it.

$3^2 + y^2 = 25$

$y^2 = 16 \rightarrow y = 4$



2) Find eqn of $\boxed{\text{the tan. line}}$ to the graph at $\boxed{x=3}$ in the $\boxed{\text{first quadrant.}}$

$m_{\text{tan. line}} = \left.\frac{dy}{dx}\right|_{(3,4)}$

$x^2 + y^2 = 25$

$2x + 2y\,y' = 0$

$y' = \frac{-x}{y}$

$= \frac{-3}{4}$

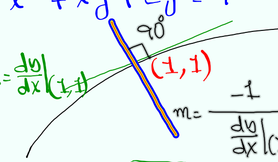
$y - y_1 = m(x - x_1)$

$y - 4 = \frac{-3}{4}(x - 3) \rightarrow y = \frac{-3}{4}x + \frac{9}{4} + 4$

$\boxed{y = \frac{-3}{4}x + \frac{25}{4}}$

Jul 7-8:55 AM

Find eqn of the normal line to the graph of $x^3 + xy + 2y^2 = 4$ at the point $(1,1)$.



$m = \left.\frac{dy}{dx}\right|_{(1,1)}$

$m = \frac{-1}{\left.\frac{dy}{dx}\right|_{(1,1)}}$

$x^3 + xy + 2y^2 = 4$

$1^3 + 1\cdot 1 + 2\cdot 1^2 \overset{?}{=} 4$

$1 + 1 + 2 \overset{?}{=} 4$

$4 \overset{?}{=} 4$ ✓

$x^3 + \boxed{xy} + \boxed{2y^2} = 4$

$3x^2 + 1\cdot y + x\cdot\frac{dy}{dx} + 2\cdot 2y\cdot\frac{dy}{dx} = 0$

$3x^2 + y + x\boxed{\frac{dy}{dx}} + 4y\boxed{\frac{dy}{dx}} = 0$

$(x + 4y)\frac{dy}{dx} = -3x^2 - y$ $\qquad \frac{dy}{dx} = \frac{-3x^2 - y}{x + 4y}$

$m_{\text{Normal line}} = \frac{-1}{\frac{-4}{5}} = \boxed{\frac{5}{4}}$ $\qquad \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-3 - 1}{1 + 4} = \frac{-4}{5}$

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{5}{4}(x - 1) \longrightarrow y = \frac{5}{4}x - \frac{5}{4} + 1$

$\boxed{y = \frac{5}{4}x - \frac{1}{4}}$

Jul 7-9:03 AM

find $\frac{dy}{dx}$

1) $\boxed{x^2 \sin y} + \boxed{y^2 \cos x} = \boxed{y}$

$$2x\sin y + x^2\cos y\,\frac{dy}{dx} + 2y\frac{dy}{dx}\cos x + y^2\cdot -\sin x = \frac{dy}{dx}$$

$$\left(x^2\cos y + 2y\cos x - 1\right)\frac{dy}{dx} = y^2\sin x - 2x\sin y$$

$$\boxed{\frac{dy}{dx} = \frac{y^2\sin x - 2x\sin y}{x^2\cos y + 2y\cos x - 1}}$$

Jul 7-9:13 AM

$y = x^4 - 8x^2 + 8$

1) Find y' $\quad y' = 4x^3 - 16x$

2) Solve $y' = 0$ $\quad 4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x = 0 \qquad x^2 - 4 = 0$

$\boxed{x = 0} \qquad \boxed{x = \pm 2}$

3) Find y'' $\quad y'' = 12x^2 - 16$

4) Solve $y'' = 0$ $\quad 12x^2 - 16 = 0$

$x^2 = \frac{16}{12} \qquad x^2 = \frac{4}{3}$

$x = \pm\sqrt{\frac{4}{3}}$

$= \pm\frac{2}{\sqrt{3}} = \boxed{\pm\frac{2\sqrt{3}}{3}}$

Jul 7-9:48 AM

$f(x) = \frac{x}{x^2-9} = x(x^2-9)^{-1}$

1) Give domain $x^2-9 \neq 0$ $x^2 \neq 9$ $x \neq \pm 3$

$(-\infty,-3) \cup (-3,3) \cup (3,\infty)$

-3 3

2) Find $f'(x)$ $f'(x) = \frac{1(x^2-9) - x \cdot 2x}{(x^2-9)^2} = \frac{x^2-9-2x^2}{(x^2-9)^2}$

$= \frac{-x^2-9}{(x^2-9)^2} = \frac{-(x^2+9)}{(x^2-9)^2}$

3) Find all x such that $f'(x)=0$ or $f'(x)$ undefined

$f'(x)=0$

$-(x^2+9)=0$

No real Soln.

$f'(x)$ is undefined

$(x^2-9)^2=0$

$x^2-9=0$

$\boxed{x=\pm 3}$

Jul 7-9:53 AM

$f(x) = x + \cos x$ $[0, 2\pi]$

1) $f'(x)$ $f'(x) = 1 - \sin x$

2) Solve $f'(x)=0$ $1-\sin x = 0 \rightarrow \sin x = 1$

$\boxed{x = \frac{\pi}{2}}$

3) $f''(x)$ $f''(x) = -\cos x$

4) Solve $f''(x)=0$ $-\cos x = 0$ $\cos x = 0$

$\boxed{x=\frac{\pi}{2}}, \boxed{x=\frac{3\pi}{2}}$

Jul 7-10:01 AM

Find $\frac{dy}{dx}$

$$\tan\boxed{\frac{x}{y}} = x + y$$

$$\sec^2\frac{x}{y}\cdot\frac{1\cdot y - x\cdot\frac{dy}{dx}}{y^2} = 1 + \frac{dy}{dx}$$

LCD $= y^2$

$$\sec^2\frac{x}{y}\left(y - x\frac{dy}{dx}\right) = y^2 + y^2\frac{dy}{dx}$$

$$y\sec^2\frac{x}{y} - x\sec^2\frac{x}{y}\frac{dy}{dx} = y^2 + y^2\frac{dy}{dx}$$

$$y\sec^2\frac{x}{y} - y^2 = \left(x\sec^2\frac{x}{y} + y^2\right)\frac{dy}{dx}$$

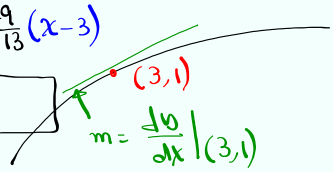
$$\frac{dy}{dx} = \frac{y\sec^2\frac{x}{y} - y^2}{x\sec^2\frac{x}{y} + y^2}$$

Jul 7-10:06 AM

Find eqn of the tan. line to the graph of $2(x^2+y^2)^2 = 25(x^2-y^2)$ at $(3,1)$.

$y - 1 = \frac{-9}{13}(x-3)$

$\boxed{y=\quad}$



$m = \frac{dy}{dx}\Big|_{(3,1)}$

$2(x^2+y^2)^2 = 25(x^2-y^2)$

$2(3^2+1^2)^2 = 25(3^2-1^2)$

$2\cdot 10^2 = 25(8)$

$200 = 200$ ✓

$$2\cdot 2(x^2+y^2)^1\cdot\left(2x + 2y\frac{dy}{dx}\right) = 25\left(2x - 2y\frac{dy}{dx}\right)$$

$$4(3^2+1^2)(2\cdot 3 + 2\cdot 1\cdot m) = 25(2\cdot 3 - 2\cdot 1\cdot m)$$

$$40(6+2m) = 25(6-2m)$$

Divide by 5

$8(6+2m) = 5(6-2m)$

$48 + 16m = 30 - 10m$

$16m + 10m = 30 - 48$

$26m = -18$

$m = \frac{-18}{26}$

$\boxed{m = \frac{-9}{13}}$

Jul 7-10:15 AM

Show $y = cx^2$ and $x^2+2y^2=K$ are orthogonal curves.

$y = cx^2$ $\qquad$ $x^2+2y^2=K$

$y' = 2cx$ $\qquad$ $2x+4yy'=0$

$4yy' = -2x$

Show product of derivatives is -1.

$y' = \frac{-2x}{4y} = \frac{-x}{2y}$

$\cancel{2}cx \cdot \frac{-x}{\cancel{2}y} = -\frac{cx^2}{y} = -\frac{y}{y} = \boxed{-1}$ ✓

orthogontal Curves.

Jul 7-10:26 AM

Estimate $\sqrt[3]{1001}$ $\qquad$ Calc. → ≈ 10.003 ✓

$\sqrt[3]{1001} \approx \sqrt[3]{1000} = 10$

Linear Approximation

$f(x) = \sqrt[3]{x}$

$a = 1000$

$f(x) \approx f(a) + f'(a)(x-a)$

$\sqrt[3]{x} \approx \sqrt[3]{1000} + \frac{1}{3(\sqrt[3]{1000})^2}(x-1000)$

$f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$\sqrt[3]{x} \approx 10 + \frac{1}{300}(x-1000)$

$= \frac{1}{3x^{2/3}} = \frac{1}{3(\sqrt[3]{x})^2}$

for $x = 1001$

$\sqrt[3]{1001} \approx 10 + \frac{1}{300}(1001-1000)$

$= 10 + \frac{1}{300} = \frac{3001}{300}$

≈ 10.003 ✓

Jul 7-10:32 AM

Looking ahead

$y^2 = 2x + 1 \quad , \quad x = x(t) \; , \; y = y(t)$

Find $\frac{dy}{dt}$ when $x = 4$, $\boxed{y = 3}$ and $\boxed{\frac{dx}{dt} = 2}$.

Take derivative of both sides with respect to t.

$$\frac{d}{dt}\left[y^2\right] = \frac{d}{dt}\left[2x + 1\right]$$

$$2y \frac{dy}{dt} = 2\frac{dx}{dt} + 0$$

$$2 \cdot 3 \frac{dy}{dt} = 2 \cdot 2 \qquad \boxed{\frac{dy}{dt} = \frac{2}{3}}$$

Jul 7-10:40 AM

$4x^2 + 9y^2 = 36$

Find $\frac{dx}{dt}$ if $\frac{dy}{dt} = \frac{1}{3}$, $x = 2$, and $y = \frac{2\sqrt{5}}{3}$

$$\frac{d}{dt}\left[4x^2\right] + \frac{d}{dt}\left[9y^2\right] = \frac{d}{dt}\left[36\right]$$

$$4 \cdot 2x \cdot \boxed{\frac{dx}{dt}} + 9 \cdot 2y \cdot \frac{dy}{dt} = 0$$

$$8 \cdot 2 \frac{dx}{dt} + \overset{2}{\cancel{18}} \cdot \frac{2\sqrt{5}}{\cancel{3}} \cdot \frac{1}{\cancel{3}} = 0$$

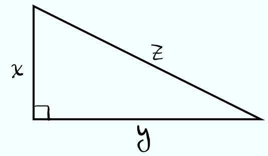
$$16 \frac{dx}{dt} + 4\sqrt{5} = 0$$

$$\frac{dx}{dt} = \frac{-4\sqrt{5}}{16} \qquad \boxed{\frac{dx}{dt} = \frac{-\sqrt{5}}{4}}$$

Jul 7-10:46 AM

Consider the drawing below where

$x = x(t)$, $y = y(t)$, $z = z(t)$



1) write an equation that has x, y, and z in it.

Pythagorean Thrm

$$x^2 + y^2 = z^2$$

2) find z when $x = 3$, and $y = 4$.

$$3^2 + 4^2 = z^2$$

$$z^2 = 25 \qquad \boxed{z = 5}$$

3) find $\frac{dz}{dt}$ when $x = 3$, $\frac{dx}{dt} = 2$, $y = 4$, $\frac{dy}{dt} = 5$.

$$x^2 + y^2 = z^2$$

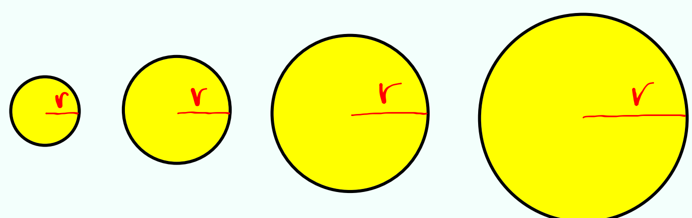
$$\frac{d}{dt}[x^2] + \frac{d}{dt}[y^2] = \frac{d}{dt}[z^2]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$3 \cdot 2 + 4 \cdot 5 = 5\frac{dz}{dt}$$

$$26 = 5\frac{dz}{dt} \qquad \boxed{\frac{dz}{dt} = \frac{26}{5}}$$

Jul 7-10:53 AM



As radius increases, Area increases.

Circle $\longrightarrow$ $A = \pi r^2$

Find changes in the area when $\boxed{r = 5}$ and $\boxed{\frac{dr}{dt} = 4}$.

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$$

$$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt}$$

$$= 2\pi r\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(5)(4)$$

$$= \boxed{40\pi}$$

Jul 7-11:02 AM